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GENERALIZED SYNDROME DECODING PROBLEM AND ITS APPLICATION TO POST-QUANTUM CRYPTOGRAPHY

PhD thesis in theoretical computer science

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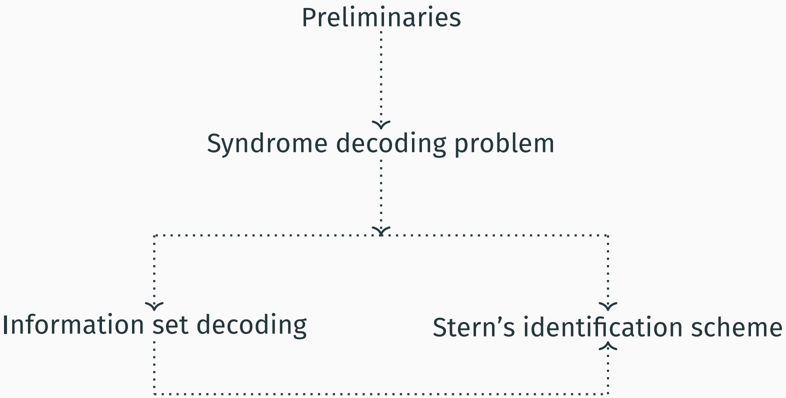
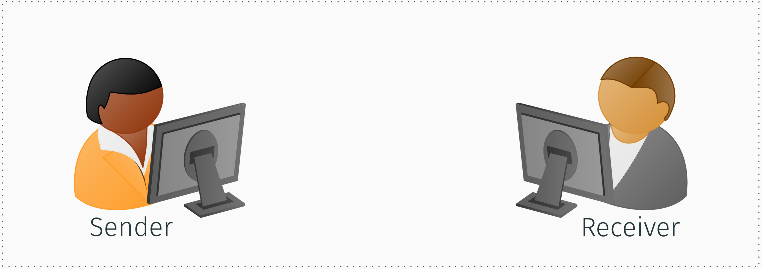
OUTLINE


Preliminaries
Syndrome decoding problem
Information set decoding
Stern's identification scheme

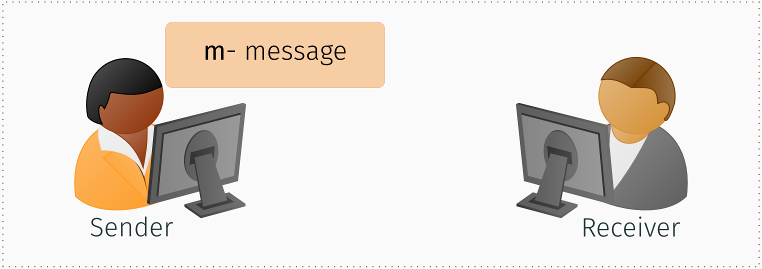
PRELIMINARIES

Coding theory



Basic setting

CODING THEORY



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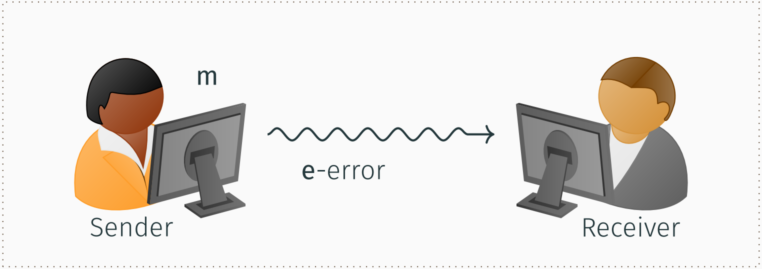

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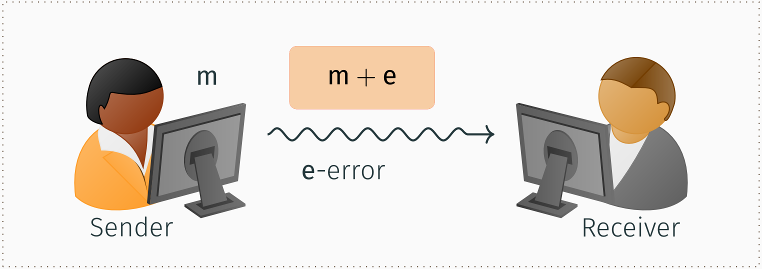

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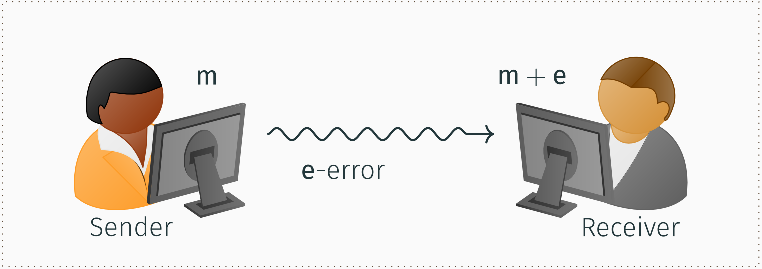

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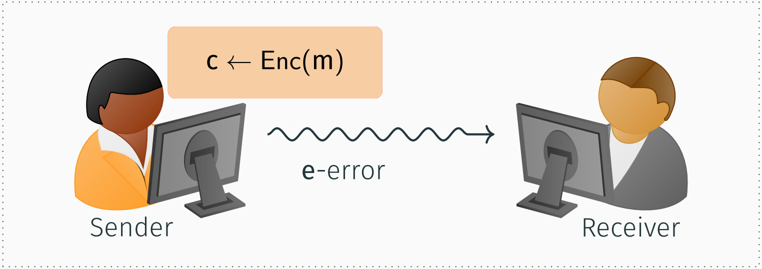

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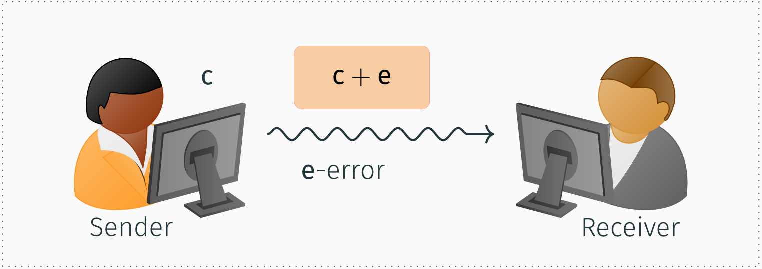

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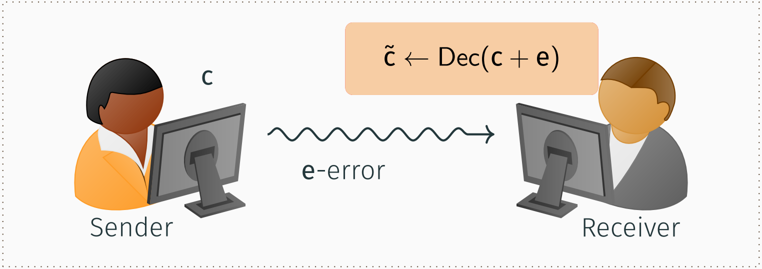

Basic setting

Message encoding

- message $\mathbf{m}$ of length k, with symbols from alphabet of size q

$$\rightarrow \boxed{\mathbf{m} \in \mathbb{F}_q^k}$$

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- encoding algorithm Enc that maps message into codeword

$$\rightarrow \mathsf{Enc} : \mathbb{F}_q^k \rightarrow \mathbb{F}_q^n$$

Message encoding

Encoding is commonly defined via a **generator matrix**, $\mathsf{G} \in \mathbb{F}_q^{k \times n}$:

$$\forall \mathsf{m} \in \mathbb{F}_q^k, \quad \mathsf{Enc}(\mathsf{m}) := \mathsf{m}^\mathsf{T}\mathsf{G}.$$

Message encoding

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$$\forall \mathbf{m} \in \mathbb{F}_q^k, \quad \mathsf{Enc}(\mathbf{m}) := \mathbf{m}^\top \mathsf{G}.$$

A **code**, $\mathcal{C}$, is then defined as:

$$\mathcal{C} := \{\mathbf{c} \in \mathbb{F}_q^n \mid (\exists \mathbf{m} \in \mathbb{F}_q^k)\ \mathbf{c} = \mathsf{Enc}(\mathbf{m})\}.$$

Message encoding

Equivalently, linear code can be defined via a **parity check matrix**, $\mathsf{H} \in \mathbb{F}_q^{(n-k)\times n}$, which is a matrix of maximal rank that satisfies:

$$\mathsf{H}\mathsf{G}^\top = 0.$$

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$$\rightarrow \boxed{\tilde{\mathbf{c}} \in \mathbb{F}_q^n}$$

- decoding algorithm Dec that maps noisy codeword, $\tilde{\mathbf{c}}$, into codeword $\mathbf{c} \in \mathcal{C}$

$$\rightarrow \boxed{\mathsf{Dec} : \mathbb{F}_q^n \rightarrow \mathbb{F}_q^n}$$

Hamming weight

Hamming distance, $\mathrm{dist}_H(\cdot)$

$$\forall \mathbf{c} = (c_0, \ldots, c_{n-1}) \in \mathbb{F}_q^n, \quad \forall \tilde{\mathbf{c}} = (\tilde{c}_0, \ldots, \tilde{c}_{n-1}) \in \mathbb{F}_q^n,$$
$$\mathrm{dist}_H(\mathbf{c}, \tilde{\mathbf{c}}) = |\{i \in [n] : c_i \neq \tilde{c}_i\}|$$

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Hamming weight, $\mathrm{wt}_H(\cdot)$

$$\forall \mathbf{e} = (e_1, \ldots, e_n) \in \mathbb{F}_q^n, \quad \mathrm{wt}_H(\mathbf{e}) := \mathrm{dist}_H(\mathbf{e}, 0).$$

Message decoding

Decoding methods:

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- **minimum distance decoding** - given the noisy codeword, $\tilde{\mathbf{c}}$, find the codeword, $\mathbf{c}$, at smallest **Hamming distance**;

Message decoding

Decoding methods:

- **minimum distance decoding** - given the noisy codeword, $\tilde{\mathbf{c}}$, find the codeword, $\mathbf{c}$, at smallest **Hamming distance**;
- **syndrome decoding**: calculate the syndrome, $\mathbf{s} \in \mathbb{F}_q^{n-k}$, defined as:

$$\mathbf{s} := \mathbf{H}\tilde{\mathbf{c}} = \mathbf{H}(\mathbf{c} + \mathbf{e}) = \mathbf{H}\mathbf{e},$$

  find the error, $\mathbf{e}$, of the smallest **Hamming weight** that corresponds to $\mathbf{s}$.

SYNDROME DECODING PROBLEM (SDP)

Computational problem derived from the **syndrome decoding method**.

Syndrome Decoding Problem, SDP

Input – A parity check matrix $\mathbf{H} \in \mathbb{F}_q^{(n-k)\times n}$, a syndrome $\mathbf{s} \in \mathbb{F}_q^{n-k}$, and a weight $w \in \mathbb{N}$.

Goal – Find an error $\mathbf{e} \in \mathbb{F}_q^n$ such that $\mathbf{s} = \mathbf{H}\mathbf{e}$ and $\mathrm{wt}(\mathbf{e}) = w$.

Syndrome Decoding Problem

An **NP-complete** problem.[1]

[1] Elwyn R. Berlekamp, Robert J. McEliece, and Henk C. A. van Tilborg. "On the inherent intractability of certain coding problems (Corresp.)". In: (1978), pp. 384–386. DOI: 10.1109/TIT.1978.1055873.

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An **NP-complete** problem.

For conveniently chosen parameters, the problem is exponentially hard for the best known **classical** and **quantum** algorithms.

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$\Rightarrow$ It is believed to be **post-quantum**.

Syndrome Decoding Problem

An **NP-complete** problem.

For conveniently chosen parameters, the problem is exponentially hard for the best known **classical** and **quantum** algorithms.

Used as basis of different cryptographic protocols.[1],[2]

[1] R. J. McEliece. "A Public-Key Cryptosystem Based On Algebraic Coding Theory". In: Deep Space Network Progress Report 44 (Jan. 1978), pp. 114–116.

[2] Jacques Stern. "A New Identification Scheme Based on Syndrome Decoding". In: 1993, pp. 13–21. DOI: 10.1007/3-540-48329-2_2.

Generalized Syndrome Decoding Problem, GSDP

Input – A parity check matrix $\mathbf{H} \in \mathbb{F}_q^{(n-k)\times n}$, a syndrome $\mathbf{s} \in \mathbb{F}_q^{n-k}$, and a weight $w \in \mathbb{N}$.

Goal – Find an error $\mathbf{e} \in \mathbb{F}_q^n$ such that $\mathbf{s} = \mathbf{H}\mathbf{e}$ and $\mathrm{wt}_M(\mathbf{e}) = w$.

Elementwise weight functions, $\mathrm{wt}_M : \mathbb{F}_q^n \to \mathbb{N}$

$$\forall \mathbf{e} = (e_0, \ldots, e_{n-1}) \in \mathbb{F}_q^n, \quad \mathrm{wt}_M(\mathbf{e}) = \sum_i \mathrm{dist}(e_i, 0),$$

where $\mathrm{dist} : \mathbb{F}_q \times \mathbb{F}_q \to \mathbb{N}$ is a distance function (metric).

Examples of elementwise weight functions

Hamming distance, $\mathrm{dist}_H(\cdot,\cdot)$

$$\forall a, b \in \mathbb{F}_q, \quad \mathrm{dist}_H(a,b) = \begin{cases} 0, & a = b \\ 1, & \text{otherwise} \end{cases} .$$

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Examples of elementwise weight functions

Lee distance, $\mathrm{dist}_L(\cdot,\cdot)$

$$\forall a, b \in \mathbb{F}_q, \quad \mathrm{dist}_L(a, b) = \min(|a - b|, q - |a - b|).$$

Examples of elementwise weight functions

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Lee weight, $\mathrm{wt}_L(\cdot)$

$$\forall \mathbf{e} = (e_1, \ldots, e_n) \in \mathbb{F}_q^n, \quad \mathrm{wt}_L(\mathbf{e}) = \sum_i \mathrm{wt}_l(e_i).$$

Our goals

Estimate the **asymptotic complexity** of the generalized syndrome decoding problem.

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Apply the generalized syndrome decoding problem to a **concrete cryptographic setting**.

INFORMATION SET DECODING (ISD)

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The best generic algorithms for solving the syndrome decoding problem.

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The best generic algorithms for solving the syndrome decoding problem.

Exploit the **linear structure** of the linear codes.

Algorithm Information set decoding

Input : $\mathbf{H} \in \mathbb{F}_q^{(n-k)\times n}$, $\mathbf{s} \in \mathbb{F}_q^{n-k}$, $w, d, l \in \mathbb{N}$.

Output: $\mathbf{e} \in \mathbb{F}_q^{n}$ s.t. $\mathbf{He} = \mathbf{s}$ and $\mathrm{wt}_M(\mathbf{e}) = w$.

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permutation step: permutes columns of $\mathbf{H}$

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- **test step:** checks if any solution from the list $\mathcal{L}$ yields a solution to the original problem

end

return e

partial Gaussian elimination step: given permuted $\mathbf{H}$ and $\mathbf{s}$, as well as d and l, creates a GSDP subinstance

$$\mathbf{U}\mathbf{H}_\pi = \begin{pmatrix} \mathbf{I} & \mathbf{H}_1 \\ \mathbf{0} & \mathbf{H}_2 \end{pmatrix}, \quad \mathbf{U}\mathbf{s} = \begin{pmatrix} \mathbf{s}_1 \\ \mathbf{s}_2 \end{pmatrix} \quad \Rightarrow \quad \begin{cases} \mathbf{e}_1 + \mathbf{H}_1\mathbf{e}_2 = \mathbf{s}_1 \\ \boxed{\mathbf{H}_2\mathbf{e}_2 = \mathbf{s}_2} \end{cases}.$$

where $\mathbf{e}_{\pi^{-1}} = \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}$ is a permuted solution to the problem

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test step: for each $\mathbf{e}_2 \in \mathcal{L}$, calculate $\mathbf{e}_1 \leftarrow \mathbf{s}_1 - \mathbf{H}_1\mathbf{e}_2$ and verify if $\mathrm{wt}_M(\mathbf{e}_1) = w - d$

Different (classical) ISD variants

ISD algorithms differ primarily in the last two steps of the algorithm, namely, **Multi-solution SDP step** and **Test step**.

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Prange's algorithm[a] takes $\mathbf{e} \leftarrow \begin{pmatrix} s_1 \\ 0 \end{pmatrix}$ and verify if $\mathrm{wt}_M(\mathbf{e}) = w$.

[a] E. Prange. "The use of information sets in decoding cyclic codes". In: IRE Transactions on Information Theory (1962), pp. 5–9. DOI: 10.1109/TIT.1962.1057777.

Different (classical) ISD variants

ISD algorithms differ primarily in the last two steps of the algorithm, namely, **Multi-solution SDP step** and **Test step**.

Lee-Brickel's algorithm[a], for each $\mathbf{e}_2$ of weight d, calculates

$$\mathbf{e}_1 \leftarrow \mathbf{s}_1 - \mathbf{H}_1\mathbf{e}_2$$

and verify if $\mathrm{wt}_M(\mathbf{e}_1) = w - d$.

[a] Pil Joong Lee and Ernest F. Brickell. "An Observation on the Security of McEliece's Public-Key Cryptosystem". In: 1988.

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ISD algorithms differ primarily in the last two steps of the algorithm, namely, **Multi-solution SDP step** and **Test step**.

Stern's/Dumer's algorithm[a], merges two lists of elements of weight $d/2$ to obtain a list, $\mathcal{L}$, of elements of weight d.

For each $\mathbf{e}_2$ in $\mathcal{L}$, the algorithm calculates

$$\mathbf{e}_1 \leftarrow \mathbf{s}_1 - \mathbf{H}_1\mathbf{e}_2$$

and verify if $\mathrm{wt}_M(\mathbf{e}_1) = w - d$.

[a] Jacques Stern. "A New Identification Scheme Based on Syndrome Decoding". In: 1993, pp. 13–21. DOI: 10.1007/3-540-48329-2_2.

Different (classical) ISD variants

ISD algorithms differ primarily in the last two steps of the algorithm, namely, **Multi-solution SDP step** and **Test step**.

Wagner's algorithm[a], for a chosen a, merges 2^a lists of elements of weight $d/2^a$ to obtain a list, $\mathcal{L}$, of elements of weight d.

For each $\mathbf{e}_2$ in $\mathcal{L}$, the algorithm calculates

$$\mathbf{e}_1 \leftarrow \mathbf{s}_1 - \mathbf{H}_1\mathbf{e}_2$$

and verify if $\mathrm{wt}_M(\mathbf{e}_1) = w - d$.

[a] Jacques Stern. "A New Identification Scheme Based on Syndrome Decoding". In: 1993, pp. 13–21. DOI: 10.1007/3-540-48329-2_2.

Our contributions: Part 1

Generalized ISD framework solving the generalized syndrome decoding problem.

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Derivation of a hybrid quantum-classical ISD algorithm.

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Derivation of a hybrid quantum-classical ISD algorithm.

Numerical results on the asymptotic analysis of the running time of ISD when solving GSDP over q-ary Hamming and Lee weight.

Classical ISD algorithms

The choice of l, d, and a give us the following algorithms:

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- $l = 0, \ d = 0, \ a = 1 \Rightarrow$ Prange's algorithm[3];

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- $l \geq 0,\ d \geq 0,\ a = 1 \Rightarrow$ Stern's/Dumer's algorithm[3];

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- $l = 0,\ d \geq 0,\ a = 1 \Rightarrow$ Lee-Brickel's algorithm ;
- $l \geq 0,\ d \geq 0,\ a = 1 \Rightarrow$ Stern's/Dumer's algorithm ;
- $l \geq 0,\ d \geq 0,\ a \geq 1 \Rightarrow$ Wagner's algorithm[3].

[3] David A. Wagner. “A Generalized Birthday Problem”. In: ed. by Moti Yung. 2002, pp. 288–303. DOI: 10.1007/3-540-45708-9_19.

Input : $\mathbf{H} \in \mathbb{F}_q^{(n-k)\times n}$, $\quad \mathbf{s} \in \mathbb{F}_q^{n-k}$, $\quad w, d, l \in \mathbb{N}$.
Output: $\mathbf{e} \in \mathbb{F}_q^n$ s.t. $\mathbf{He} = \mathbf{s}$ and $\mathrm{wt}_M(\mathbf{e}) = w$.

while e is not found **do**

- **permutation step:** permutes columns of **H** — poly(n)
- **partial Gaussian elimination step:** given permuted **H** and **s**, as well as d and l, creates a GSDP subinstance — poly(n)
- **multi-solution GSDP step:** returns a list $\mathcal{L}$ of solution to the GSDP subinstance — T_{SUB}
- **test step:** checks if any solution from the list $\mathcal{L}$ yields a solution to the original problem — $|\mathcal{L}|\,\mathrm{poly}(n)$

end
return e

Running time of classical ISD algorithms

$$T_C(n, l, d, a) = \frac{\text{poly}(n) + T_{SUB}(n, l, d, a) + |\mathcal{L}|\,\text{poly}(n)}{p(n, l, d, a)},$$

where $p(\cdot, \cdot, \cdot, \cdot)$ is the probability of success in the test step.

Probability of success

$$p(n, l, d, a) = \min\left(1, \frac{\mathrm{surf}_M(q, n-k-l, w-d)}{\max\left(q^{n-k}, \mathrm{surf}_M(q, n, w)\right) q^{-l}} |\mathcal{L}|\right).$$

where

- $\mathrm{surf}_M(q, n, w)$ is the surface area of a sphere of radius w in $\mathbb{F}_q^n$,
- $\mathrm{surf}_M(q, n-k-l, w-d)$ is the surface area of a sphere of radius $w-d$ in $\mathbb{F}_q^{n-k-l}$.

Probability of success

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where

- $\mathrm{surf}_M(q,n,w)$ is the surface area of a sphere of radius w in $\mathbb{F}_q^n$,
- $\mathrm{surf}_M(q,n-k-l,w-d)$ is the surface area of a sphere of radius $w-d$ in $\mathbb{F}_q^{n-k-l}$.

Major obstacle: calculating the surface area of a sphere in a vector space endowed with arbitrary elementwise weight function.

Quantum Wagner's algorithm

A hybrid classical-quantum algorithm was obtained as a combination of:

- classical Wagner's algorithm,
- Grover's search[4],
- amplitude amplification[5].

[4] Lov K. Grover. "A Fast Quantum Mechanical Algorithm for Database Search". In: 1996, pp. 212–219. DOI: 10.1145/237814.237866.

[5] Gilles Brassard, Peter Høyer, et al. Quantum amplitude amplification and estimation. 2002.

Quantum Wagner's algorithm

Definition: Grover's algorithm[4]

Let $f : \{0,1\}^n \to \{0,1\}$ has an efficient classical description.

Grover's algorithm can find i such $f(i) = 1$ in time $O(poly(n)2^{n/2})$ if such an i exists and output 'no solution' otherwise.

[4] Lov K. Grover. "A Fast Quantum Mechanical Algorithm for Database Search". In: 1996, pp. 212–219. DOI: 10.1145/237814.237866.

Quantum Wagner's algorithm

Definition: Amplitude amplification[4]

Let $f : \{0,1\}^n \rightarrow \{0,1\}$ has an efficient classical description.

Consider an algorithm $\mathcal{A}$ that outputs i such that $f(i) = 1$ with probability p, and $f(i) = 0$ with probability $1 - p$.

Using amplitude amplification, one can find i such that $f(i) = 1$ by making $O(\frac{1}{\sqrt{p}})$ calls to $\mathcal{A}$.

[4] Gilles Brassard and Peter Hoyer. "An Exact Quantum Polynomial-Time Algorithm for Simon's Problem". In: 1997, pp. 12–23. DOI: 10.1109/ISTCS.1997.595153.

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The difference appears only in the **multi-solution GSDP step** and **test step**:

- in the **multi-solution GSDP step**, the algorithm returns a **description of a function** $f : [|\mathcal{L}|] \to \mathbb{F}_q^n$
- in the **test step** the algorithm checks if any output of $f(\cdot)$ yields a solution to the original problem using **Grover’s search**

Quantum Wagner's algorithm

Input : $H \in \mathbb{F}_q^{(n-k)\times n}$, $s \in \mathbb{F}_q^{n-k}$, $w, d, l, a \in \mathbb{N}$.
Output : $e \in \mathbb{F}_q^n$ s.t. $He = s$ and $wt_M(e) = w$.

while e is not found **do**

- **permutation and partial Gaussian elimination step:** permute columns of H and create a GSDP subinstance — $\text{poly}(n)$
- **multi-solution GSDP step:** returns a **description of** $f : [|\mathcal{L}|] \to \mathbb{F}_q^n$ that outputs solutions to the GSDP subinstance — T_{SUB}
- **test step:** using **Grover's search**, checks if any output of $f(\cdot)$ yields a solution to the original problem — $\sqrt{|\mathcal{L}|}\,\text{poly}(n)$

end
return e

Quantum Wagner's algorithm

Running time

$$T_Q(n, l, d, a) = \frac{\mathrm{poly}(n) + T_{SUB}(n, l, d, a) + \sqrt{|\mathcal{L}|}\,\mathrm{poly}(n)}{\sqrt{p}(n, l, d, a)},$$

where p is the probability of success in the test step.

Numerical results

The asymptotic running time is evaluated when parameters l, d, and a are optimized to yield the shortest running time.

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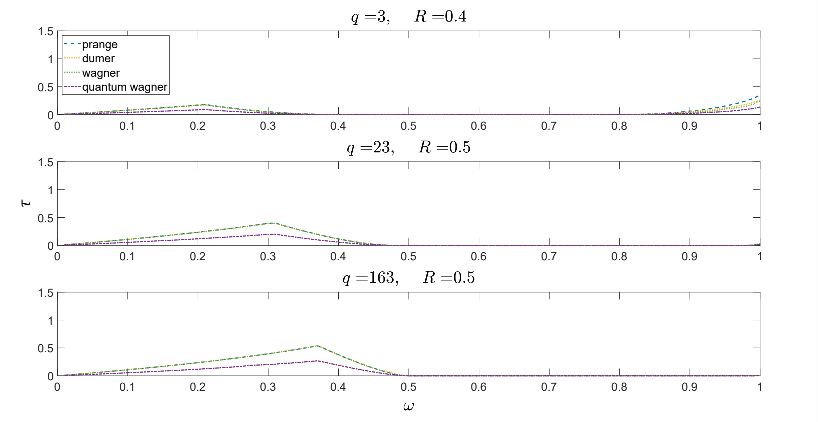
Exponent of the asymptotic running time, τ

$$\tau(q, R, \omega) = \lim_{n\to\infty} \frac{1}{n} \log_2 T(n),$$

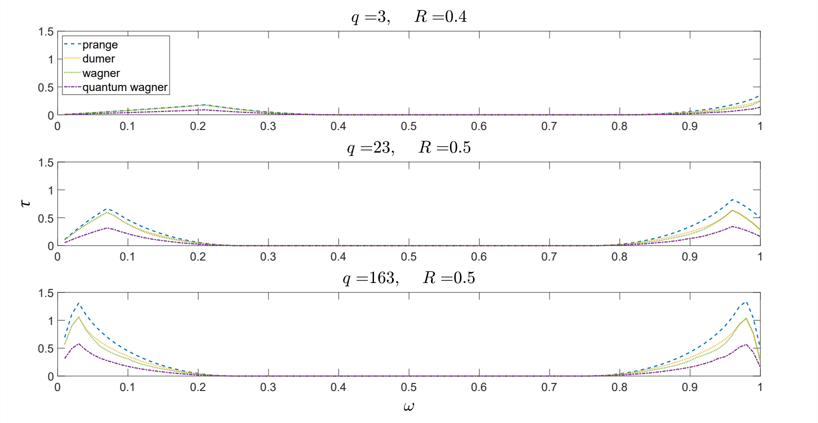
where $R := \frac{k}{n}$ and $\omega := \frac{w}{n}$.

Numerical results



Hamming weight setting: $\tau(q, R, \omega) = \lim_{n\to\infty} \frac{1}{n} \log_2 T$, $R := \frac{k}{n}$, and $\omega : \frac{w}{n}$

Numerical results



Lee weight setting: $\tau(q, R, \omega) = \lim_{n\to\infty} \frac{1}{n} \log_2 T$, $R := \frac{k}{n}$, and $\omega : \frac{w}{n}$

Summary of the first part

The asymptotic complexity of the **hardest instances** of GSDP problem is in the **Lee weight** setting is at least as long as in the Hamming weight case.

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The asymptotic complexity of the **hardest instances** of GSDP problem is in the **Lee weight** setting is at least as long as in the Hamming weight case.

For the **quantum setting**, our algorithms have almost a **quadratic improvement** over the classical setting.

The **GSDP problem** remains **exponentially har**d for conveniently chosen parameters both in the classical and quantum setting.

STERN'S IDENTIFICATION PROTOCOL

Stern's identification protocol[4]

Belongs to the class of so-called **sigma** or **three-round** protocols.

[4] Jacques Stern. "A New Identification Scheme Based on Syndrome Decoding". In: 1993, pp. 13–21. DOI: 10.1007/3-540-48329-2_2.

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The protocol is unbroken for almost 30 years now, but suffers from rather **high communication costs**.

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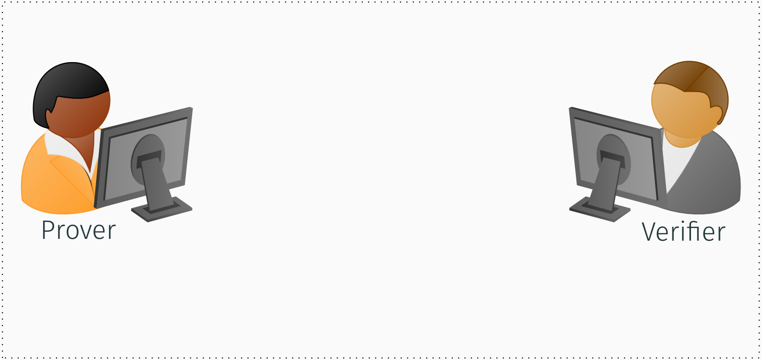
Sigma (3-round) protocol

A two-party, public-key protocol.



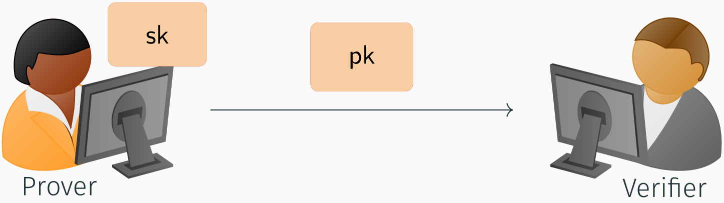
Sigma (3-round) protocol

Prover

$\mathsf{sk}, \mathsf{pk} \leftarrow \mathsf{Keygen}(1^\lambda)$

Key generation

Sigma (3-round) protocol



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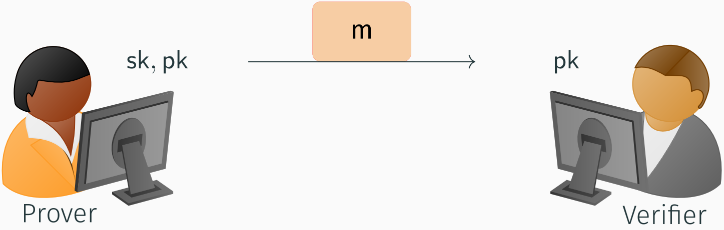
Key generation

SIGMA (3-ROUND) PROTOCOL

$(\mathbf{m}, \mathsf{init}) \leftarrow P_1(\mathsf{sk})$

Interaction

Sigma (3-round) protocol



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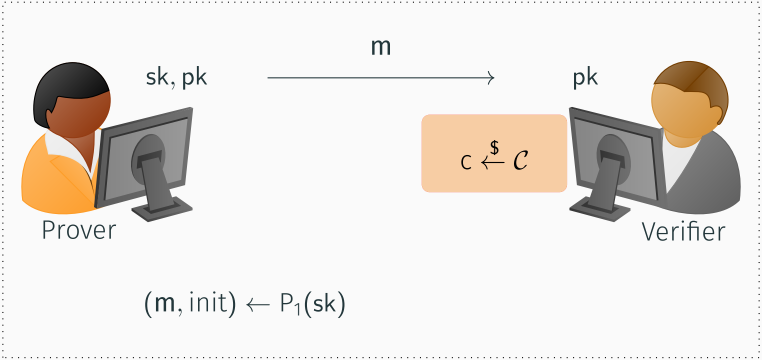

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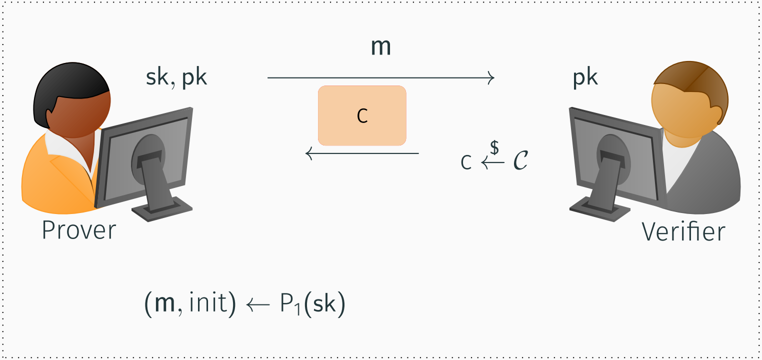

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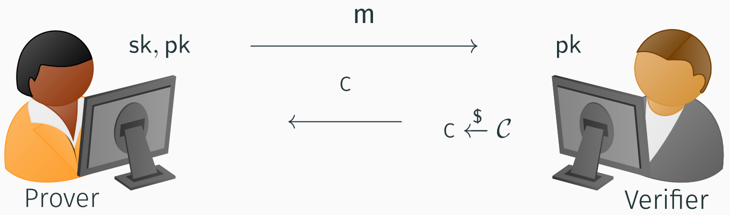
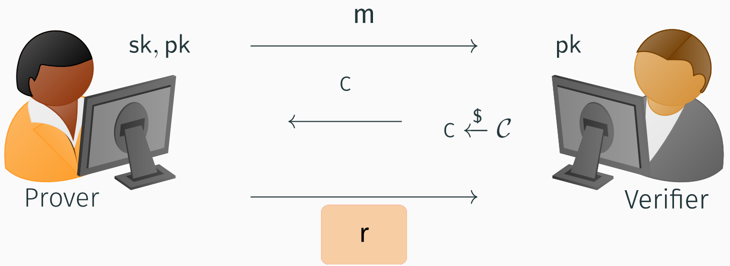


$(\mathbf{m}, \text{init}) \leftarrow P_1(\text{sk})$

$\mathbf{r} \leftarrow P_2(\text{sk}, \text{init}, c)$

Interaction

Sigma (3-round) protocol



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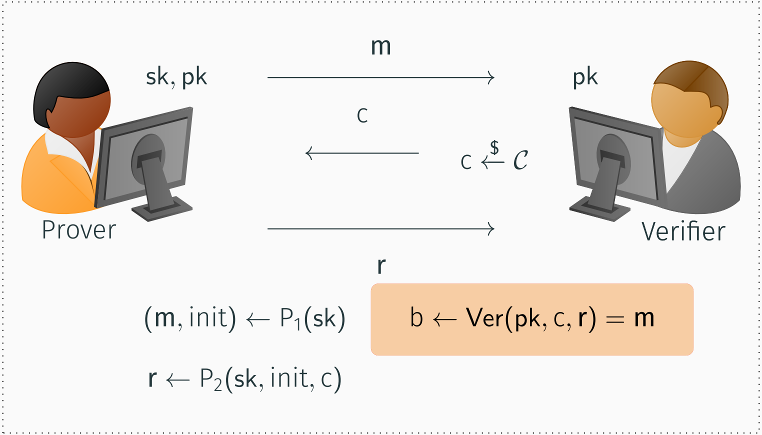
SIGMA (3-ROUND) PROTOCOL



Verifying

Sigma (3-round) protocol

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Basic properties:

- **completness**: honest prover needs to be able to convince verifier it knows sk;
- **soundness**: dishonest prover is not able to convince verifier it knows sk with probability 1;
- **zero-knolwedge**: communication reveals only if prover knows sk and nothing else.

Stern's identification protocol

$$H \xleftarrow{\$} \mathbb{F}_q^{(n-k)\times n}, \quad e \xleftarrow{\$} \mathbb{F}_q^n, \quad s = He$$

Key generation

Stern's identification protocol

$$H \xleftarrow{\$} \mathbb{F}_q^{(n-k)\times n}, \quad e \xleftarrow{\$} \mathbb{F}_q^n, \quad s \leftarrow He$$

$$pk \leftarrow (H, s), \quad sk \leftarrow e$$

Key generation

Stern's identification protocol

$$\pi \xleftarrow{\$} \mathrm{Perm}[n],\ \mathbf{y} \xleftarrow{\$} \mathbb{F}_q^n,\ \mathbf{t} \leftarrow \mathbf{H}\mathbf{y},$$

Interaction

Stern's identification protocol

sk

Prover

pk

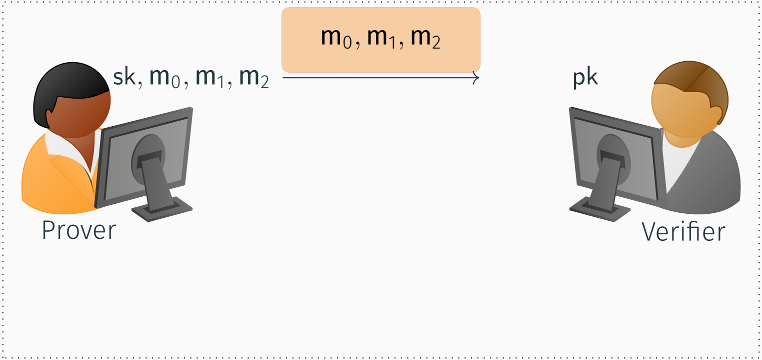
Verifier

$$\pi \xleftarrow{\$} \mathrm{Perm}[n],\ \mathbf{y} \xleftarrow{\$} \mathbb{F}_q^n,\ \mathbf{t} \leftarrow \mathbf{H}\mathbf{y},$$

$$m_0 \leftarrow \mathcal{H}(\pi, \mathbf{t}),\ m_1 \leftarrow \mathcal{H}(\pi(\mathbf{y})),\ m_2 \leftarrow \mathcal{H}(\pi(\mathbf{y} + \mathbf{e}))$$

Interaction

Stern's identification protocol



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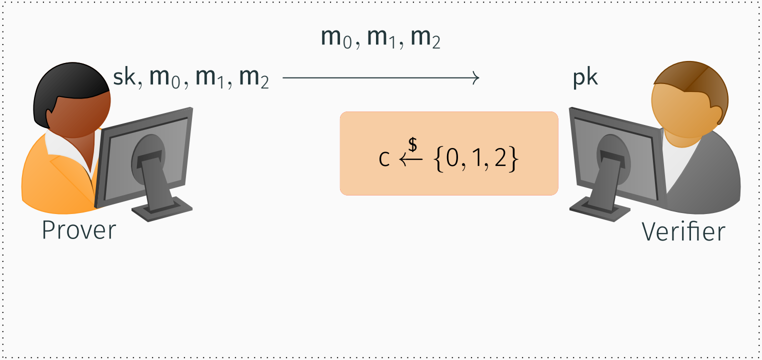

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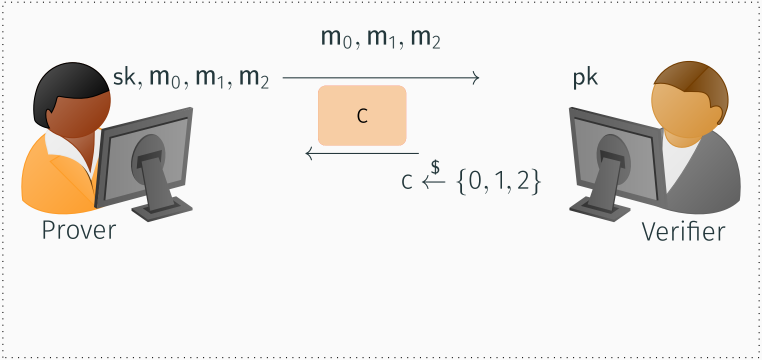

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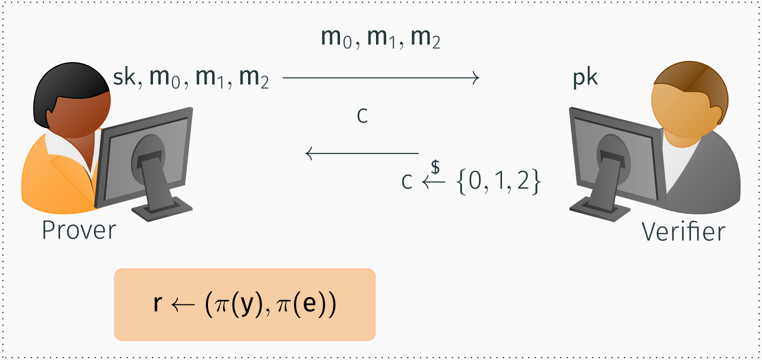
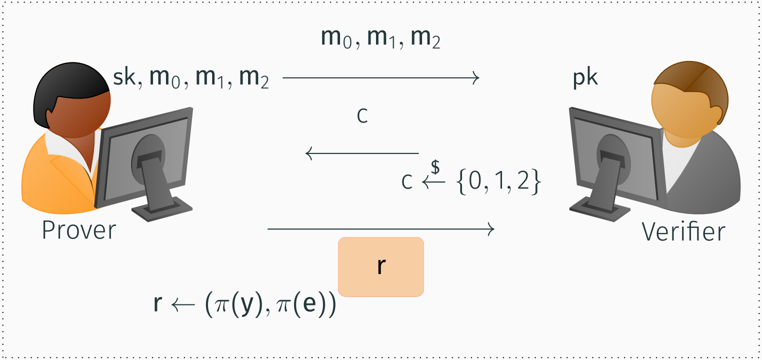


Interaction: case $c = 0$

Stern's identification protocol



Interaction

Stern's identification protocol

m_0, m_1, m_2

Prover: sk, m_0, m_1, m_2 $\longrightarrow$ Verifier: pk

c $\longleftarrow$ $c \xleftarrow{\$} \{0,1,2\}$

r $\longrightarrow$

Prover

Verifier

$r \leftarrow (\pi(y), \pi(e))$

$b \leftarrow \quad m_1 \stackrel{?}{=} H(\pi(y)) \wedge$

$m_2 \stackrel{?}{=} H(\pi(y) + \pi(e)) \;\wedge\; wt_M(\pi(e)) \stackrel{?}{=} w$

Verifying: case c = 0

Stern's identification protocol

Basic properties:

- the scheme is **complete**;
- it is **sound**, with **soundness error** of 2/3;
- it is proven to be **honest verifier zero-knowledge** in the random oracle model.

Stern's identification protocol

Basic properties:

- the scheme is **complete**;
- it is **sound**, with **soundness error** of 2/3;
- it is proven to be **honest verifier zero-knowledge**.

Soundness error can be reduced arbitrarily close to zero by repeating the protocol r times.

Stern's identification protocol

Major drawback: high communication costs (order of 100 kB).

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$\rightarrow$ Reduction of communication cost can be achieved using **pseudo random generators** and **deterministic commitments**.

Our contributions[5]

A **honest verifier zero-knowledge** variant of Stern's identification scheme adapted to the **generalized syndrome decoding problem**.

[5]André Chailloux and Simona Etinski. On the (In)security of optimized Stern-like signature schemes. Cryptology ePrint Archive, Paper 2021/552. 2022.

Our contributions[5]

A **honest verifier zero-knowledge** variant of Stern's identification scheme adapted to the **generalized syndrome decoding problem**.

A proof that using **deterministic commitments** in combination **pseudo random generated** random vectors is secure.

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Numerical results

Obtained for concrete parameters of GSDP that guarantee that the analyzed algorithms run in $2^{128} \Rightarrow 128$ bits of security.

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The optimized scheme is constructed using **deterministic commitments** in combination with **pseudo-random generators**.

Numerical results

q	Non-optimized scheme		Optimized scheme	
	wt_H	wt_L	wt_H	wt_L
2	253.05	253.05	26.21	26.21
3	116.54	116.54	21.81	21.81
5	138.54	95.48	27.62	21.41
7	126.47	90.94	28.29	22.71
13	113.23	79.27	29.38	23.29

Table: Communication cost of non-optimized and optimized schemes

Summary of the second part

Communication cost can be significantly reduced by using **deterministic commitments** in combination with the **pseudo-random generation**.

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Communication cost can be significantly reduced by using **deterministic commitments** in combination with the **pseudo-random generation**.

Without loss in security, **additional reduction** can be obtained by replacing the original SDP with it's **generalized version over Lee weight**.

Future directions

Generalize the asymptotic analysis to the ISD algorithms based on **representation techniques**, **nearest neighbour search**, and **statistical decoding**.

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Generalize the asymptotic analysis to the ISD algorithms based on **representation techniques**, **nearest neighbour search**, and **statistical decoding**.

Apply more advanced communication reduction techniques such as **shared permutations**, **"MPC in the head"**, use **quasi-cyclic matrices**.

Merci pour votre attention !

Thank you for your attention!

Hvala vam na pažnji!