SIEVING FOR CODES: FROM GJN TO HASH-BASED AND RPC-BASED APPROACHES

February 2024, ATTACC workshop, Germany

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based on joint work with: Léo Ducas (CWI, LEI), Andre Esser (TII), and Elena Kirshanova (TII, IKBFU)

Motivation: Sieving is a well-known and widely used technique in the lattice-based cryptography.

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The idea of adapting the sieving to **information set decoding** framework was introduced in [GJN23][1].

[1] Qian Guo, Thomas Johansson, and Vu Nguyen. A New Sieving-Style Information-Set Decoding Algorithm. 2023.

Sieving ISD framework

ISD

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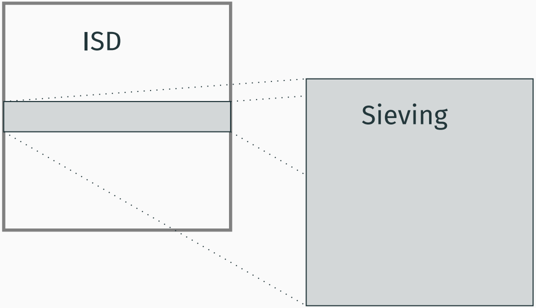

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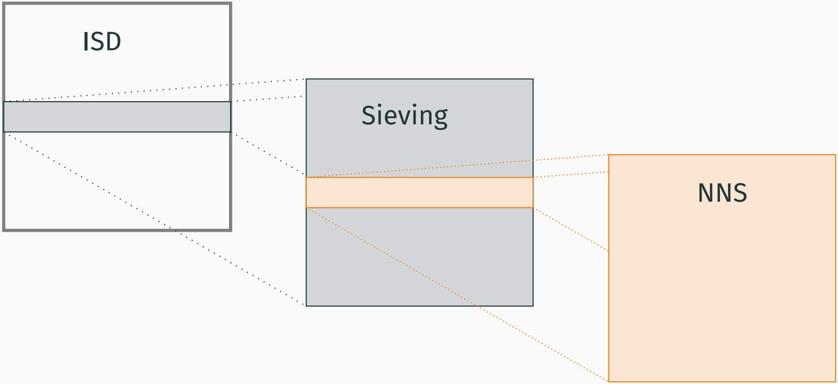

SIEVING ALGORITHM

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- $\mathcal{C} \subseteq \mathbb{F}_2^n$ - an $[n, k]$ binary linear code.

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    end

    Discard some elements if $|\mathcal{L}'| > N$ and set $\mathcal{L}' \leftarrow \mathcal{L}$.

  end

end

return $\mathcal{L}$

Sieving

The **running time** and the **memory**:

$$T_{\text{SIEVING}} = \tilde{\mathcal{O}}(T_{\text{NNS}}), \quad M_{\text{SIEVING}} = \tilde{\mathcal{O}}(M_{\text{NNS}}).$$

Remarks

Heuristics: The input list elements at any step of the sieving algorithm behave like **uniformly random** and **independent** vectors from the sphere $\mathcal{S}_w^n$.

Remarks

Choice of N: We choose N such that there exist N distinct codewords of weight w in $\mathcal{C}$ and that we maintain the list size in each iteration, namely

$$\frac{c\binom{n}{w}}{\binom{w}{w/2}\binom{n-w}{w/2}} \leq N \leq \binom{n}{w} \cdot 2^{k-n}.$$

NEAR NEIGHBOR SEARCH ALGORITHMS

Previous work

[MO15][2], [BM18][3], etc. and Kévin Carrier's thesis[4] explored **near neighbor search** in the coding setting.

[2] Alexander May and Ilya Ozerov. "On Computing Nearest Neighbors with Applications to Decoding of Binary Linear Codes". In: 2015.

[3] Leif Both and Alexander May. "Decoding Linear Codes with High Error Rate and Its Impact for LPN Security". In: ed. by Tanja Lange and Rainer Steinwandt. 2018.

[4] Kévin Carrier. "Recherche de Presque-Collisions pour le Décodage et la Reconnaissance de Codes Correcteurs. (Near-collisions finding problem for decoding and recognition of error correcting codes)". PhD thesis. 2020.

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In the lattice-based setting, **sieving** was successfully combined with **locality-sensitive hashing (filtering)** introduced in [BDGL15][2].

[2]Anja Becker et al. New directions in nearest neighbor searching with applications to lattice sieving. 2015.

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High-level idea

If two vectors overlap in α positions, they are more likely to be close in space (aka these are ”near neighbors”).

Algorithm Near Neighbour Search

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  for ValidFilters($\mathcal{C}_f, \alpha, \mathbf{x}$) **do**

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for x ∈ L do
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        add x to B_c
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for $x \in \mathcal{L}$ **do**
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for $x \in \mathcal{L}$ **do**
  for $\mathbf{c} \in$ ValidFilters$(\mathcal{C}_f, \alpha, \mathbf{x})$, $\mathbf{y} \in \mathcal{B}_c$ **do**
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        if |x + y| = w then
            add (x, y) to P
        end
    end
end
return P
```

Near Neighbor Search

The **running time**:

$$T_{\text{NNS}} = \tilde{\mathcal{O}}(N \cdot T_{\text{VALIDFILTERS}}) + \tilde{\mathcal{O}}(N \cdot \mathbb{E}(\text{VALIDFILTERS}) \cdot \mathbb{E}(\mathcal{B})).$$

The **memory**:

$$M_{\text{NNS}} = \tilde{\mathcal{O}}(N \cdot \mathbb{E}(\text{VALIDFILTERS})).$$

GJN, HASH-BASED AND RPC-BASED

Guo, Johansson and Nguyen [GJN] approach[3]

Main idea

For any $\mathbf{x}, \mathbf{y} \in \mathcal{S}_w^n$ satisfying $|\mathbf{x} + \mathbf{y}| = w$, there exists $\mathbf{c} \in \mathcal{S}_{w/2}^n$ such that $|\mathbf{x} \wedge \mathbf{c}| = |\mathbf{y} \wedge \mathbf{c}| = w/2$.

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*Initially, the approach was not presented in the locality-sensitive filtering fashion, yet it aligns with the framework.

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Parameters:

$$\mathcal{C}_{\mathsf{f}} = \mathcal{S}^{\mathsf{n}}_{\mathsf{w}/2}, \quad \alpha = \mathsf{w}/2.$$

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Valid Filters Subroutine

For each $\mathbf{x} \in \mathcal{L}$, returns all $\mathbf{c} \in \mathcal{S}^n_{w/2}$ such that $|\mathbf{x} \wedge \mathbf{c}| = w/2$.

Coded hashing approach (HASH)[3]

Parameters

$$\mathcal{C}_f = \mathcal{S}^n_\alpha \cap \mathcal{C}_\mathcal{H}, \quad \alpha \leq w/2,$$

where $\mathcal{C}_\mathcal{H}$ is $[n, n-r]$ binary linear code.

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[3] Léo Ducas et al. Asymptotics and Improvements of Sieving for Codes. 2023.

Random product codes approach (RPC)[4]

Parameters:

$$\mathcal{C}_{\mathcal{H}}^{(i)} \subseteq \mathcal{S}_{v/t}^{n/t}, \quad \mathcal{C}_{\mathcal{H}} = \mathcal{C}_{\mathcal{H}}^{(1)} \times \cdots \times \mathcal{C}_{\mathcal{H}}^{(t)}, \quad \alpha, v \leq w/2 \text{ - to be optimized.}$$

Valid Filters Subroutine

For each $\mathbf{x} = (\mathbf{x}^{(1)}, \ldots \mathbf{x}^{(t)}) \in \mathcal{L}$, returns all $\mathbf{c} = (\mathbf{c}^{(1)}, \ldots \mathbf{c}^{(t)}) \in \mathcal{S}_v^n \cap \mathcal{C}_{\mathcal{H}}$ such that

$$|\mathbf{x}^{(i)} \wedge \mathbf{c}^{(i)}| = \alpha/t \text{ for all } i \in \{1, \ldots, t\}.$$

[4] Léo Ducas et al. Asymptotics and Improvements of Sieving for Codes. 2023.

Memory optimal versions (HASH and RPC memo-opt)[5]

High-level idea

We interleave the bucketing and the checking phase.

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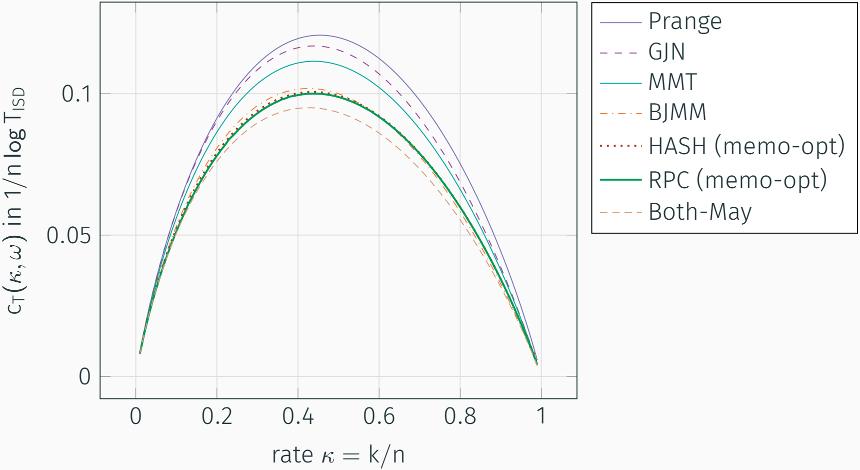
Memory optimal approach

The initial set of filters contains $|\mathcal{C}_f|/2^d$ centers but we repeat the algorithm 2^d times.

[5] Léo Ducas et al. Asymptotics and Improvements of Sieving for Codes. 2023.

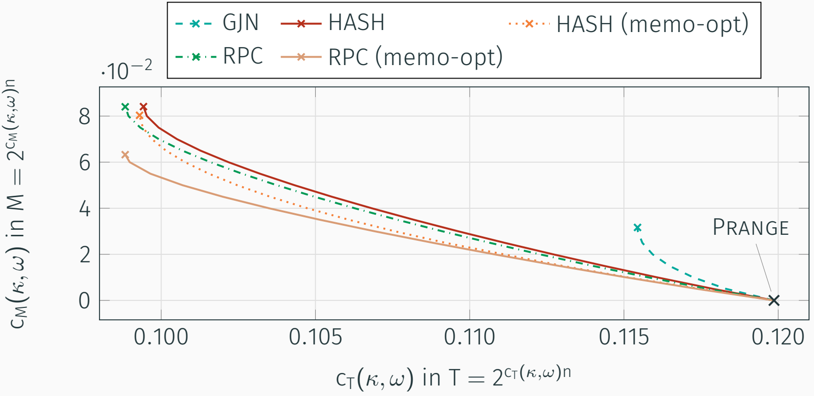
NUMERICAL RESULTS



Runtime exponent for different ISD and SievingISD variants.

Type	Algorithm	κ	$c_T(\kappa,\omega)$	$c_M(\kappa,\omega)$
SievingISD	GJN	0.44	0.1169	0.0279
	HASH	0.44	0.1007	0.0849
	HASH memo-opt	0.44	0.1007	0.0830
	RPC	0.44	0.1001	0.0852
	RPC memo-opt	0.44	0.1001	0.0636
Conventional ISD	Prange	0.45	0.1207	0.0000
	MMT	0.45	0.1116	0.0541
	BJMM	0.43	0.1020	0.0728
	Both-May	0.42	0.0951	0.0754

Table: Worst case running time $2^{c_T(\kappa,\omega)n}$ and corresponding memory usage $2^{c_M(\kappa,\omega)n}$ for different ISD algorithms.



Time-memory trade-off curves of different SievingISD instantiations, for $\kappa = 0.5$ and $\omega = H^{-1}(0.5)$.

CONCLUDING REMARKS

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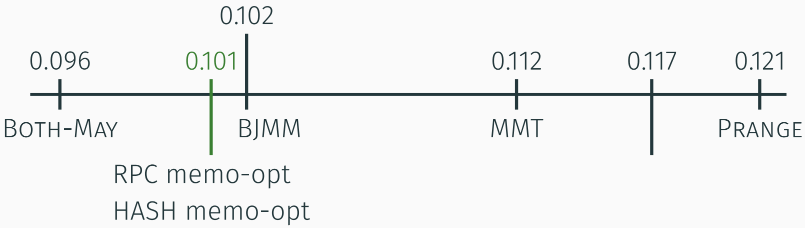
→ For the worst case, the efficiency is comparable with BJMM.



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A new alignment of the lattice-based and code-based framework (equivalent to [BDGL15]).

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Instead of shortening vectors, we iteratively reduce the coset size till we find vectors in the code but we keep their length unchanged.

Open questions

How applicable it is?

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Is it inherently different from the other ISD algorithms?

Thank you for your attention!

eprint

GitHub repo

MCCL fork